

Superspace Unitary Operator in QED with Dirac and Complex Scalar Fields: Superfield Approach

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Abstract: We exploit the strength of the superspace (SUSP) unitary operator to obtain the results of the application of the horizontality condition (HC) within the framework of augmented version of superfield formalism that is applied to the *interacting* systems of Abelian 1-form gauge theories where the $U(1)$ Abelian 1-form gauge field couples to the Dirac and complex scalar fields in the physical four $(3 + 1)$ -dimensions of spacetime. These interacting theories are generalized onto a $(4, 2)$ -dimensional supermanifold that is parametrized by the four $(3 + 1)$ -dimensional (4D) spacetime variables and a pair of Grassmannian variables. To derive the (anti-)BRST symmetries for the *matter* fields, we impose the gauge invariant restrictions (GIRs) on the superfields defined on the $(4, 2)$ -dimensional supermanifold. We discuss various outcomes that emerge out from our knowledge of the SUSP unitary operator and its hermitian conjugate. The latter operator is derived without imposing any operation of hermitian conjugation on the parameters and fields of our theory from *outside*. This is an interesting observation in our present investigation.

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1 Introduction

One of the most elegant and geometrically intuitive approaches to the p -form ($p = 1, 2, 3, \dots$) gauge theories, described within the framework of Becchi-Rouet-Stora-Tyutin (BRST) formalism, is the superfield approach (see, e.g. [1-5]). In particular, in refs. [1-3], it has been shown that one can derive the proper (i.e. off-shell nilpotent and absolutely anticommuting) (anti-) BRST symmetry transformations for the non-Abelian 1-form gauge and corresponding (anti-)ghost fields by exploiting the potential and power of the horizontality condition (HC) where the super curvature 2-form ($\tilde{F}^{(2)}$), defined on the $(D, 2)$ -dimensional supermanifold, is equated with the ordinary curvature 2-form ($F^{(2)}$) defined on the D -dimensional Minkowskian flat spacetime manifold. However, the above superfield formalism [1-5] does *not* shed any light on the derivation of (anti-)BRST symmetry transformations associated with the matter fields of a given interacting non-Abelian 1-form gauge theory where there is a coupling between the gauge field and the Noether conserved current constructed with the matter fields (as far as the minimal interaction in a given gauge theory is concerned).

In a set of papers [6-9], the above superfield formalism [1-5] has been consistently generalized so as to derive the proper (anti-)BRST symmetry transformations for the *matter* fields (in addition to the gauge and (anti-)ghost fields where the input from the outcomes of the HC plays an important role (see, e.g. [8,9] for details)). The generalized version of superfield formalism (where the HC and gauge invariant restrictions (GIRs) are exploited together) has been christened as the augmented version of superfield formalism. In Ref. [1-3], a superspace (SUSP) unitary operator has been intelligently chosen which provides the (anti-)BRST symmetry transformations for the matter, (anti-)ghost and gauge fields where the gauge group structure of the specific gauge theory is very elegantly maintained. However, the explicit mathematical derivation of this operator has not been provided in these seminal works [1-3]. It would be a nice idea to exploit the key concepts of augmented superfield formalism to derive this SUSP unitary operator clearly.

The purpose of our present paper is to derive the above SUSP unitary operator elegantly and explicitly in the case of interacting Abelian 1-form gauge theories with Dirac and complex scalar fields. In this connection, first of all, we exploit the potential of the HC to derive the (anti-)BRST symmetry transformations for the Abelian 1-form gauge and corresponding (anti-)ghost fields. Subsequently, we utilize *this* result to derive the (anti-)BRST symmetry transformations for the matter fields (i.e. Dirac and complex scalar fields) to obtain the explicit form of the SUSP unitary operator where the SUSP $U(1)$ gauge group structure is maintained. We exploit the explicit mathematical form of this operator to derive the results of HC and prove the reasons behind the imposition of HC in the superfield approach to BRST formalism. This is one of the highlights of our present investigation.

One of the key consequences of the SUSP unitary operator is that the matter field transforms (e.g. $\Psi^{(g)}(x, \theta, \bar{\theta}) = U(x, \theta, \bar{\theta}) \psi(x)$) in such a manner that the SUSP $U(1)$ gauge group structure is respected in the transformation space. As a result, one can define the covariant derivative which would also transform in exactly the same manner (i.e. $\tilde{D}\Psi^{(g)}(x, \theta, \bar{\theta}) = U(x, \theta, \bar{\theta}) D\psi(x)$). This, in turn, defines the transformation of the supercurvature 2-form (i.e. $\tilde{F}^{(2)} = U(x, \theta, \bar{\theta}) F^{(2)} U^\dagger(x, \theta, \bar{\theta})$) which leads to the derivation of the HC (i.e. $\tilde{F}^{(2)} = F^{(2)}$) (because the SUSP unitary operator $U(x, \theta, \bar{\theta})$ is Abelian in nature and $U^\dagger U = U U^\dagger = 1$). Thus, we obtain an alternative to the HC in the language of the SUSP

unitary operator and, in some sense, we provide the proof for the validity of the HC (i.e. $\tilde{F}^{(2)} = F^{(2)}$) in the context of superfield approach to any arbitrary D -dimensional Abelian gauge theory (described within the framework of BRST formalism).

Our present endeavor is motivated by the following factors. First, the SUSP unitary operator $U(x, \theta, \bar{\theta})$ has been judiciously chosen in [1-3]. However, it has *not* been theoretically derived. We have accomplished this goal in our present endeavor. Second, the accurate derivation of this SUSP operator provides the *proof* behind the imposition of the HC in the context of superfield approach to BRST formalism. Third, the $U(1)$ group structure appears very naturally in the theory due to the transformation property (e.g. $\Psi^{(g)}(x, \theta, \bar{\theta}) = U(x, \theta, \bar{\theta}) \psi(x)$, etc.). Fourth, the results of HC are reproduced by using the SUSP unitary operator which provides, in some sense, an alternative to it. Finally, our present endeavor for the Abelian theory is our first modest step towards our main goal of obtaining the SUSP unitary operator for the non-Abelian theory.

Our present paper is organized as follows. In Sec. 2, we discuss the importance of HC in the derivation of complete set of proper (anti-)BRST symmetry transformations for the gauge and (anti-)ghost fields of this theory. Our Sec. 3 lays emphasis on the derivation of (anti-)BRST symmetries for the *matter* fields and the SUSP unitary operator (which is responsible for the shift transformations along the Grassmannian directions of the supermanifold). In Sec. 4, we derive the (anti-)BRST symmetry transformations for the gauge and (anti-)ghost fields by exploiting the strength of the SUSP unitary operator (which is equivalent to the application of HC). Finally, we make some concluding remarks and point out a few future directions for further investigations in Sec. 5.

2 Preliminaries: HC and (anti-)BRST symmetries

We start off with the following (anti-)BRST invariant Lagrangian density $\mathcal{L}_B^{(D)}$ for the interacting four $(3 + 1)$ -dimensional (4D) $U(1)$ gauge theory with Dirac fields ψ and $\bar{\psi}$ (with mass m and electric charge e) as*

$$\mathcal{L}_B^{(D)} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi} (i\gamma^\mu D_\mu - m) \psi + B(\partial \cdot A) + \frac{B^2}{2} - i \partial_\mu \bar{C} \partial^\mu C, \quad (1)$$

where the covariant derivative $D_\mu \psi = \partial_\mu \psi + i e A_\mu \psi$ and the 2-form $F^{(2)} = dA^{(1)}$ defines the curvature tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ for the 1-form $A^{(1)} = dx^\mu A_\mu$ connection A_μ where $d = dx^\mu \partial_\mu$ (with $d^2 = 0$) is the exterior derivative. In the above, B field is the Nakanishi-Lautrup auxiliary field which is used for the linearization of the gauge-fixing term: $[-\frac{1}{2}(\partial \cdot A)^2]$ and $(\bar{C})C$ are the fermionic ($C^2 = \bar{C}^2 = 0, C\bar{C} + \bar{C}C = 0$) (anti-)ghost fields. The above

*We adopt here the convention and notations such that the background 4D Minkowskian flat spacetime metric $(\eta_{\mu\nu})$ has the signatures $(+1, -1, -1, -1)$ so that $(\partial \cdot A) = \partial_\mu A^\mu \equiv \eta_{\mu\nu} \partial^\mu A^\nu = \partial_0 A_0 - \partial_i A_i$ where the Greek indices $\mu, \nu, \lambda, \dots = 0, 1, 2, 3$ correspond to the spacetime directions and the Latin indices $i, j, k, \dots = 1, 2, 3$ stand for the space directions only.

Lagrangian density respects the following (anti-) BRST symmetry transformations[†] [8]

$$\begin{aligned}
s_b A_\mu &= \partial_\mu C, & s_b C &= 0, & s_b \bar{C} &= i B, \\
s_b B &= 0, & s_b \psi &= -i e C \psi, & s_b \bar{\psi} &= -i e \bar{\psi} C, \\
s_{ab} A_\mu &= \partial_\mu \bar{C}, & s_{ab} \bar{C} &= 0, & s_{ab} C &= -i B, \\
s_{ab} B &= 0, & s_{ab} \psi &= -i e \bar{C} \psi, & s_{ab} \bar{\psi} &= -i e \bar{\psi} \bar{C}.
\end{aligned} \tag{2}$$

It can be shown that the above transformations are off-shell nilpotent ($s_{(a)b}^2 = 0$) of order two and absolutely anticommuting ($s_b s_{ab} + s_{ab} s_b = 0$) in nature.

The 4D (anti-)BRST invariant Lagrangian density $\mathcal{L}_B^{(C)}$ for the complex scalar fields $\varphi(x)$ and $\varphi^*(x)$ (with mass m and electric charge e) is (see, e.g. [9])

$$\mathcal{L}_B^{(C)} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + (D_\mu \varphi)^* (D^\mu \varphi) - m^2 \varphi^* \varphi + B (\partial \cdot A) + \frac{B^2}{2} - i \partial_\mu \bar{C} \partial^\mu C, \tag{3}$$

where $D_\mu \varphi = (\partial_\mu + i e A_\mu) \varphi$ and $(D_\mu \varphi)^* = (\partial_\mu - i e A_\mu) \varphi^*$ are the covariant derivatives on the fields $\varphi(x)$ and $\varphi^*(x)$ and the rest of the symbols in $\mathcal{L}_B^{(C)}$ have been explained after equation (1). It can be seen that the Lagrangian density $\mathcal{L}_B^{(C)}$ respects the following off-shell nilpotent ($s_{(a)b}^2 = 0$) (anti-)BRST symmetry transformations $s_{(a)b}$, namely;

$$\begin{aligned}
s_b A_\mu &= \partial_\mu C, & s_b C &= 0, & s_b \bar{C} &= i B, & s_b B &= 0, \\
s_b \varphi &= -i e C \varphi, & s_b \varphi^* &= +i e \varphi^* C, & s_b F_{\mu\nu} &= 0, \\
s_{ab} A_\mu &= \partial_\mu \bar{C}, & s_{ab} \bar{C} &= 0, & s_{ab} C &= -i B, & s_{ab} B &= 0, \\
s_{ab} \varphi &= -i e \bar{C} \varphi, & s_{ab} \varphi^* &= +i e \varphi^* \bar{C}, & s_{ab} F_{\mu\nu} &= 0.
\end{aligned} \tag{4}$$

We also note that s_b and s_{ab} absolutely anticommute ($s_b s_{ab} + s_{ab} s_b = 0$) with each other. Physically, the nilpotency property encapsulates the fermionic nature of (anti-)BRST symmetry transformations and the linear independence of (anti-)BRST symmetry transformations is encoded in the property of absolute anticommutativity. It is worthwhile to mention that, unlike in the case of *fermionic* Dirac fields, the complex scalar fields $\varphi(x)$ and $\varphi^*(x)$ *commute* with the (anti-)ghost fields C and \bar{C} .

To derive the proper (i.e. nilpotent and absolutely anticommuting) (anti-)BRST symmetry transformations for the gauge and (anti-)ghost fields, within the framework of superfield formalism [1-3], we apply the HC[‡] on the super 1-form ($\tilde{A}^{(1)}$), defined on the (4, 2)-dimensional supermanifold (with the help of super exterior derivative \tilde{d}) as [1-3,8,9]

$$\tilde{d} \tilde{A}^{(1)} = d A^{(1)} \iff \tilde{F}^{(2)} = F^{(2)}, \tag{5}$$

[†]We shall use, throughout the whole body of our text, the notations $s_{(a)b}$ for the continuous and infinitesimal (anti-)BRST symmetry transformations connected with the 4D *interacting* Abelian 1-form gauge theories (of the Dirac and complex scalar fields).

[‡]Physically, the HC implies that, for the free Abelian 1-form gauge theory, the gauge invariant physical electric and magnetic fields must remain independent of the Grassmannian variables of the (4, 2)-dimensional supermanifold. This is an essential requirement from the viewpoint of physics because the Grassmannian variables are merely a sort of mathematical artifacts which are useful only in the description of superspace formulation. Furthermore, these variables are *not* physically realized unlike the spacetime variables.

where $F^{(2)} = [(dx^\mu \wedge dx^\nu) / 2] F_{\mu\nu}$ is the curvature 2-form defined on the 4D ordinary space-time manifold and $\tilde{F}^{(2)} = [(dZ^M \wedge dZ^N) / 2] \tilde{F}_{MN}$ is the supercurvature 2-form defined on the (4, 2)-dimensional supermanifold. We have the following explicit generalizations, namely;

$$\begin{aligned} d &= dx^\mu \partial_\mu \longrightarrow \tilde{d} = dZ^M \partial_M = dx^\mu \partial_\mu + d\theta \partial_\theta + d\bar{\theta} \partial_{\bar{\theta}}, \\ A^{(1)} &= dx^\mu A_\mu \longrightarrow \tilde{A}^{(1)} = dZ^M A_M \equiv dx^\mu B_\mu(x, \theta, \bar{\theta}) + d\theta \bar{F}(x, \theta, \bar{\theta}) + d\bar{\theta} F(x, \theta, \bar{\theta}), \end{aligned} \quad (6)$$

where the superspace coordinate $Z^M = (x^\mu, \theta, \bar{\theta})$ and the super derivative $\partial_M = (\partial_\mu, \partial_\theta, \partial_{\bar{\theta}})$ characterize the (4, 2)-dimensional supermanifold and $A_M = (B_\mu, F, \bar{F})$ corresponds to a vector supermultiplet. Here the spacetime coordinates x^μ (with $\mu = 0, 1, 2, 3$) are the bosonic variables and $(\theta, \bar{\theta})$ is a pair of Grassmannian variables (with $\theta^2 = \bar{\theta}^2 = 0$, $\theta \bar{\theta} + \bar{\theta} \theta = 0$). The superfields $B_\mu(x, \theta, \bar{\theta})$, $F(x, \theta, \bar{\theta})$ and $\bar{F}(x, \theta, \bar{\theta})$ can be expanded along the Grassmannian directions of the (4, 2)-dimensional supermanifold as [1-3,8,9]

$$\begin{aligned} B_\mu(x, \theta, \bar{\theta}) &= A_\mu(x) + \theta \bar{R}_\mu(x) + \bar{\theta} R_\mu(x) + i \theta \bar{\theta} S_\mu(x), \\ F(x, \theta, \bar{\theta}) &= C(x) + i \theta \bar{B}_1(x) + i \bar{\theta} B_1(x) + i \theta \bar{\theta} s(x), \\ \bar{F}(x, \theta, \bar{\theta}) &= \bar{C}(x) + i \theta \bar{B}_2(x) + i \bar{\theta} B_2(x) + i \theta \bar{\theta} \bar{s}(x), \end{aligned} \quad (7)$$

which yield the basic fields (A_μ, C, \bar{C}) of our starting Lagrangian densities (1) and (3) in the limit $\theta = \bar{\theta} = 0$. In the above, the fields $(\bar{R}_\mu, R_\mu, S_\mu, \bar{B}_1, B_1, s, \bar{s}, \bar{B}_2, B_2)$ are the secondary fields which are to be determined in terms of the basic and auxiliary fields of the Lagrangian density (1). In fact, it can be explicitly checked that we obtain (see, e.g. [8,9] for details)

$$\begin{aligned} R_\mu &= \partial_\mu C, & S_\mu &= \partial_\mu B, & \bar{R}_\mu &= \partial_\mu \bar{C}, & B_1 &= s = \bar{s} = \bar{B}_2 = 0, \\ \bar{B}_1 + B_2 &= 0 & \implies & & \bar{B}_1 &= -B = -B_2, \end{aligned} \quad (8)$$

when we exploit the HC[§] (5). The substitution of (8) into (7) yields

$$\begin{aligned} B_\mu^{(h)}(x, \theta, \bar{\theta}) &= A_\mu + \theta (\partial_\mu \bar{C}) + \bar{\theta} (\partial_\mu C) + \theta \bar{\theta} (i \partial_\mu B) \\ &\equiv A_\mu + \theta (s_{ab} A_\mu) + \bar{\theta} (s_b A_\mu) + \theta \bar{\theta} (s_b s_{ab} A_\mu), \\ F^{(h)}(x, \theta, \bar{\theta}) &= C + \theta (-i B) \equiv C + \theta (s_{ab} C), \\ \bar{F}^{(h)}(x, \theta, \bar{\theta}) &= \bar{C} + \bar{\theta} (i B) \equiv \bar{C} + \bar{\theta} (s_b \bar{C}), \end{aligned} \quad (9)$$

where the superscript (h) stands for the superfields that have been derived after the application of HC. It is clear, from the above, that we have already obtained the (anti-) BRST symmetry transformations for the gauge and (anti-)ghost fields (cf. (2) and (4)) for the interacting system of the $U(1)$ Abelian 1-form gauge theories (with Dirac and complex scalar fields). It is to be noted that the (anti-)BRST transformations for the gauge and (anti-)ghost fields are the *same* for both the interacting $U(1)$ gauge theories under consideration. Furthermore, it is interesting to point out that the (anti-)BRST symmetry transformations $(s_{(a)b})$ are connected with the translational generators ∂_θ and $\partial_{\bar{\theta}}$ along the Grassmannian directions of the supermanifold by the relationships: $s_b \iff \partial_{\bar{\theta}}$, $s_{ab} \iff \partial_\theta$.

[§]The horizontality condition *physically* implies that the electric and magnetic fields of the Abelian Maxwell's theory should be independent of the presence of the Grassmannian variables in SUSY theory.

3 Gauge invariant restrictions and SUSP unitary operator: (anti-)BRST symmetries for matter fields

In our previous section, we have derived the (anti-)BRST symmetry transformations for the gauge and (anti-)ghost fields but have not discussed the (anti-)BRST symmetries associated with the Dirac fields. To obtain these symmetry transformations, we impose the following gauge invariant restriction (GIR) on the matter superfields (see, e.g. [8] for details)

$$\bar{\Psi}(x, \theta, \bar{\theta}) \tilde{D}(x, \theta, \bar{\theta}) \Psi(x, \theta, \bar{\theta}) = \bar{\psi}(x) D \psi(x), \quad (10)$$

where $D = d + i e A^{(1)}$ and $\tilde{D} = \tilde{d} + i e \tilde{A}_{(h)}^{(1)}$. The super 1-form $\tilde{A}_{(h)}^{(1)}$ connection (with $\tilde{d}\tilde{A}_{(h)}^{(1)} = dA^{(1)}$) is defined, in terms of the superfields (9), as follows

$$\tilde{A}_{(h)}^{(1)}(x, \theta, \bar{\theta}) = dx^\mu B_\mu^{(h)}(x, \theta, \bar{\theta}) + d\theta \bar{F}^{(h)}(x, \theta, \bar{\theta}) + d\bar{\theta} F^{(h)}(x, \theta, \bar{\theta}), \quad (11)$$

where the explicit expansions of $B_\mu^{(h)}(x, \theta, \bar{\theta})$, $\bar{F}^{(h)}(x, \theta, \bar{\theta})$ and $F^{(h)}(x, \theta, \bar{\theta})$ are given in (9). The matter superfields $\bar{\Psi}(x, \theta, \bar{\theta})$ and $\Psi(x, \theta, \bar{\theta})$ have the following expansions along the Grassmannian directions of the (4, 2)-dimensional supermanifold, namely;

$$\begin{aligned} \Psi(x, \theta, \bar{\theta}) &= \psi(x) + i\theta \bar{b}_1(x) + i\bar{\theta} b_1(x) + i\theta \bar{\theta} t(x), \\ \bar{\Psi}(x, \theta, \bar{\theta}) &= \bar{\psi}(x) + i\theta \bar{b}_2(x) + i\bar{\theta} b_2(x) + i\theta \bar{\theta} \bar{t}(x), \end{aligned} \quad (12)$$

where $(b_1, \bar{b}_1, b_2, \bar{b}_2, t, \bar{t})$ are the secondary fields which would be determined in terms of the basic and auxiliary fields of our present theory described by the Lagrangian density (1). In this connection, the GIR in (10) helps us to obtain the following relationships between the secondary fields and basic and auxiliary fields[¶] (see, e.g. [8] for details)

$$\begin{aligned} \bar{b}_1 &= -e \bar{C} \psi, & b_1 &= -e C \psi, & t &= -i e (B - e C \bar{C}), \\ \bar{b}_2 &= -e \bar{\psi} \bar{C}, & b_2 &= -e \bar{\psi} C, & \bar{t} &= i e (B - e \bar{C} C). \end{aligned} \quad (13)$$

The substitution of these expressions into (12) yields the following explicit expansions for the matter superfields in terms of the (anti-)BRST symmetry transformations (2):

$$\begin{aligned} \Psi^{(g)}(x, \theta, \bar{\theta}) &= \psi(x) + \theta (-i e \bar{C} \psi) + \bar{\theta} (-i e C \psi) + \theta \bar{\theta} (e (B - e C \bar{C}) \psi) \\ &\equiv \psi(x) + \theta (s_{ab} \psi) + \bar{\theta} (s_b \psi) + \theta \bar{\theta} (s_b s_{ab} \psi), \\ \bar{\Psi}^{(g)}(x, \theta, \bar{\theta}) &= \bar{\psi}(x) + \theta (-i e \bar{\psi} \bar{C}) + \bar{\theta} (-i e \bar{\psi} C) + \theta \bar{\theta} (-e \bar{\psi} (B - e \bar{C} C)) \\ &\equiv \bar{\psi}(x) + \theta (s_{ab} \bar{\psi}) + \bar{\theta} (s_b \bar{\psi}) + \theta \bar{\theta} (s_b s_{ab} \bar{\psi}), \end{aligned} \quad (14)$$

where the superscript (g) denotes the expansions of the superfields obtained after the application of the GIR (10).

[¶]We slightly differ with the relationships mentioned in [8] because there is a minor printing error in the latter as far as the relationships quoted for t and \bar{t} are concerned. The relationships in (13) are *all* correct.

We are now in the position to state the precise form of the SUSP unitary operator which transforms the ordinary Dirac matter fields $\psi(x)$ and $\bar{\psi}(x)$ to their counterparts $\Psi^{(g)}(x, \theta, \bar{\theta})$ and $\bar{\Psi}^{(g)}(x, \theta, \bar{\theta})$. In fact, using the expansions (14), it is clear that^{||}

$$\begin{aligned}\Psi^{(g)}(x, \theta, \bar{\theta}) &= [1 + \theta(-ie\bar{C}) + \bar{\theta}(-ieC) + \theta\bar{\theta}(e(B - eC\bar{C}))] \psi(x) \\ &\equiv U(x, \theta, \bar{\theta}) \psi(x), \\ \bar{\Psi}^{(g)}(x, \theta, \bar{\theta}) &= \bar{\psi}(x) [1 + \theta(ie\bar{C}) + \bar{\theta}(ieC) + \theta\bar{\theta}(-e(B - eC\bar{C}))] \\ &\equiv \bar{\psi}(x) U^\dagger(x, \theta, \bar{\theta}),\end{aligned}\tag{15}$$

where $U(x, \theta, \bar{\theta})$ and $U^\dagger(x, \theta, \bar{\theta})$ are the SUSY generators which, primarily, lead to the shift transformations along the Grassmannian directions (because $\Psi^{(g)}(x, \theta, \bar{\theta}) = U(x, \theta, \bar{\theta}) \psi(x)$ and $\bar{\Psi}^{(g)}(x, \theta, \bar{\theta}) = \bar{\psi}(x) U^\dagger(x, \theta, \bar{\theta})$). These SUSP operators (i.e. U and U^\dagger) can be expressed in the mathematically precise exponential forms as^{**}

$$\begin{aligned}U(x, \theta, \bar{\theta}) &= \exp[\theta(-ie\bar{C}) + \bar{\theta}(-ieC) + \theta\bar{\theta}(eB)], \\ U^\dagger(x, \theta, \bar{\theta}) &= \exp[\theta(ie\bar{C}) + \bar{\theta}(ieC) + \theta\bar{\theta}(-eB)],\end{aligned}\tag{16}$$

which directly establish that the SUSP operator U is unitary (i.e. $UU^\dagger = U^\dagger U = 1$). This statement can be proven to be true by using the explicit expressions for U and U^\dagger that are quoted in (15) (and that are equivalent to (16)). The crucial observation is that the SUSP operator $U(x, \theta, \bar{\theta})$ forms a $U(1)$ group in the space of transformations where the exponential form (16) of the operator $U(x, \theta, \bar{\theta})$ plays an important role. Similar statement could be made with the operator $U^\dagger(x, \theta, \bar{\theta})$, too.

To obtain the (anti-)BRST symmetry transformations associated with the complex scalar fields $\varphi(x)$ and $\varphi^*(x)$ (c.f. Eq. (4)), we impose the following gauge invariant restrictions (GIRs) on the superfields defined in the (4, 2)-dimensional supermanifold [9]

$$\begin{aligned}\Phi^*(x, \theta, \bar{\theta}) \left(\tilde{d} + ie\tilde{A}_{(h)}^{(1)} \right) \Phi(x, \theta, \bar{\theta}) &= \varphi^*(x) (d + ieA^{(1)}) \varphi(x), \\ \Phi(x, \theta, \bar{\theta}) \left(\tilde{d} - ie\tilde{A}_{(h)}^{(1)} \right) \Phi^*(x, \theta, \bar{\theta}) &= \varphi(x) (d - ieA^{(1)}) \varphi^*(x),\end{aligned}\tag{17}$$

where the superfields $\Phi(x, \theta, \bar{\theta})$ and $\Phi^*(x, \theta, \bar{\theta})$ have the expansions along the Grassmannian directions of the (4, 2)-dimensional supermanifold as [9]

$$\begin{aligned}\Phi(x, \theta, \bar{\theta}) &= \varphi(x) + i\theta\bar{f}_1(x) + i\bar{\theta}f_2(x) + i\theta\bar{\theta}b(x), \\ \Phi^*(x, \theta, \bar{\theta}) &= \varphi^*(x) + i\theta f_2^*(x) + i\bar{\theta}f_1^*(x) + i\theta\bar{\theta}b^*(x).\end{aligned}\tag{18}$$

In the above, we have the secondary fields on the r.h.s. as $(\bar{f}_1, f_1^*, f_2, f_2^*, b, b^*)$. These fields could be determined in terms of the basic and auxiliary fields of the Lagrangian density

^{||}We note that the relationship $\Psi^{(g)}(x, \theta, \bar{\theta}) = U(x, \theta, \bar{\theta}) \psi(x)$ is exactly of the same kind as the $U(1)$ gauge transformation on the Dirac field: $\psi(x) \rightarrow \psi'(x) = U(x) \psi(x)$ where the operator $U(x) = e^{-ie\alpha(x)}$ (with gauge parameter $\alpha(x)$) forms the $U(1)$ group as it satisfies *all* the group properties under product.

^{**}Under the hermitian conjugation operations: $\theta^\dagger = \mp \theta, \bar{\theta}^\dagger = \mp \bar{\theta}, C^\dagger = \pm C, \bar{C}^\dagger = \pm \bar{C}, B^\dagger = B, e^\dagger = e, i^\dagger = -i$, it can be readily checked that SUSP operators $U(x, \theta, \bar{\theta})$ and $U^\dagger(x, \theta, \bar{\theta})$ interchange with each other and the FP-ghost part (i.e. $-i\partial_\mu \bar{C} \partial^\mu C$) of the Lagrangian densities (1) and (3) remains invariant.

$\mathcal{L}_B^{(C)}$ due to the GIRs in (17). It is worthwhile to mention that the r.h.s. of (17) are gauge invariant quantities and, therefore, they are (anti-)BRST invariant, too.

In our earlier work [9], all the secondary fields of (18) have been determined in a systematic manner by exploiting the strength of GIRs in (17). The outcome is:

$$\begin{aligned} \bar{f}_1 &= -e \bar{C} \varphi, & f_2 &= -e C \varphi, & b &= -i e (B - e C \bar{C}) \varphi, \\ f_1^* &= e C \varphi^*, & f_2^* &= e \bar{C} \varphi^*, & b^* &= i e (B - e \bar{C} C) \varphi^*. \end{aligned} \quad (19)$$

The substitution of these expressions into the expansion (18) leads to the following explicit expansions in terms of the (anti-)BRST symmetry transformations (4), namely;

$$\begin{aligned} \Phi^{(g)}(x, \theta, \bar{\theta}) &= \varphi(x) + \theta (-i e \bar{C} \varphi) + \bar{\theta} (-i e C \varphi) + \theta \bar{\theta} [e(B - e C \bar{C}) \varphi] \\ &\equiv \varphi(x) + \theta (s_{ab} \varphi) + \bar{\theta} (s_b \varphi) + \theta \bar{\theta} (s_b s_{ab} \varphi), \\ \Phi^{*(g)}(x, \theta, \bar{\theta}) &= \varphi^*(x) + \theta (i e \varphi^* \bar{C}) + \bar{\theta} (i e \varphi^* C) + \theta \bar{\theta} \varphi^*(x) [-e(B - e \bar{C} C)] \\ &\equiv \varphi^*(x) + \theta (s_{ab} \varphi^*) + \bar{\theta} (s_b \varphi^*) + \theta \bar{\theta} (s_b s_{ab} \varphi^*), \end{aligned} \quad (20)$$

where the superscript (g) denotes the expansions of the superfields after the application of GIRs in (17). It is pretty obvious that the above superfields can be expressed in terms of the SUSP unitary operators $U(x, \theta, \bar{\theta})$ and $U^\dagger(x, \theta, \bar{\theta})$ exactly like (15) and (16) where $\Psi^{(g)}(x, \theta, \bar{\theta})$ and $\bar{\Psi}^{(g)}(x, \theta, \bar{\theta})$ would be replaced by the superfields $\Phi^{(g)}(x, \theta, \bar{\theta})$ and $\Phi^{*(g)}(x, \theta, \bar{\theta})$ of (20). Thus, we note that the form of the SUSP unitary operators (16) remains the *same* for both *interacting* models of QED where there is an interaction between the $U(1)$ gauge field A_μ and the Noether conserved current constructed by the Dirac fields as well as the charged complex scalar fields. This happens because of the existence of the local $U(1)$ gauge group behind the construction of both these interacting theories.

4 SUSP unitary operator and HC: salient features

As a result of the group property in the transformation space, we can define a super covariant derivative on the Dirac superfield in the following fashion

$$\begin{aligned} \psi(x) &\rightarrow \Psi(x, \theta, \bar{\theta}) = U(x, \theta, \bar{\theta}) \psi(x), \\ D\psi(x) &\rightarrow \tilde{D}\Psi(x, \theta, \bar{\theta}) = U(x, \theta, \bar{\theta}) D\psi(x), \end{aligned} \quad (21)$$

where $\tilde{D} = \tilde{d} + i e \tilde{A}_{(h)}^{(1)}(x, \theta, \bar{\theta})$ and $D = d + i e A^{(1)}(x)$. It is now crystal clear that the super 1-form connection $\tilde{A}_{(h)}^{(1)}(x, \theta, \bar{\theta})$ and the ordinary 1-form connection $A^{(1)}(x)$ are connected by the following equation due to the relationships quoted in (21), namely;

$$\begin{aligned} \tilde{A}_{(h)}^{(1)}(x, \theta, \bar{\theta}) &= U(x, \theta, \bar{\theta}) A^{(1)}(x) U^\dagger(x, \theta, \bar{\theta}) + \frac{i}{e} \left(\tilde{d} U(x, \theta, \bar{\theta}) \right) U^\dagger(x, \theta, \bar{\theta}) \\ &\equiv dx^\mu B_\mu^{(h)}(x, \theta, \bar{\theta}) + d\theta \bar{F}^{(h)}(x, \theta, \bar{\theta}) + d\bar{\theta} F^{(h)}(x, \theta, \bar{\theta}). \end{aligned} \quad (22)$$

It is evident that if we use the Abelian $U(1)$ nature of the operator $\hat{U}(x, \theta, \bar{\theta})$, the first term on the r.h.s. of (22) yields the following expression

$$U(x, \theta, \bar{\theta}) A^{(1)}(x) U^\dagger(x, \theta, \bar{\theta}) = A^{(1)}(x) \equiv dx^\mu A_\mu(x). \quad (23)$$

The second term on the r.h.s. [i.e. $+(\frac{i}{e})(\tilde{d}U)U^\dagger$] leads to the following explicit expression, namely;

$$dx^\mu [\theta(\partial_\mu \bar{C}) + \bar{\theta}(\partial_\mu C) + \theta\bar{\theta}(i\partial_\mu B)] + d\theta [\bar{C}(x) + i\bar{\theta}B(x)] + d\bar{\theta} [C(x) - i\theta B(x)], \quad (24)$$

where we have used the expansions for $\tilde{d}, U(x, \theta, \bar{\theta})$ and $U^\dagger(x, \theta, \bar{\theta})$ from equations (6) and (15). Now, it is obvious that the comparison of the coefficients of $dx^\mu, d\theta$ and $d\bar{\theta}$ leads to the derivation of (anti-)BRST symmetry transformations *exactly* in the same manner as has been done in Sec. 2 where we have exploited the HC. In other words, we obtain exactly the same expressions for the $B_\mu^{(h)}, F^{(h)}$ and $\bar{F}^{(h)}$ as has been defined in Eq. (9).

From the above discussion, we can claim that the HC used in Eq. (5) is equivalent to the relationship (22) where the SUSP unitary operator plays a decisive role. To corroborate the above claim, we note that the following property of the ordinary covariant derivatives on the Dirac field is true, namely;

$$DD\psi(x) = ieF^{(2)}\psi(x), \quad (25)$$

where the covariant derivative $D = dx^\mu(\partial_\mu + ieA_\mu)$ and $F^{(2)} = [(dx^\mu \wedge dx^\nu)/2]F_{\mu\nu}(x)$. This property can be expressed in terms of the SUSP unitary operator as:

$$DD\psi(x) \longrightarrow \tilde{D}\tilde{D}\Psi(x, \theta, \bar{\theta}) = ie\tilde{F}^{(2)}U(x, \theta, \bar{\theta})\psi(x), \quad (26)$$

where D, \tilde{D} and $\tilde{F}^{(2)}$ are defined earlier. Now, using the relationship given in (21), we obtain $\tilde{D} = UDU^\dagger$. If we substitute this value into the l.h.s. (i.e. $\tilde{D}\tilde{D}\Psi$) of (26), we obtain the following relationship

$$UDD\psi = ie\tilde{F}^{(2)}U\psi \implies \tilde{F}^{(2)} = U(x, \theta, \bar{\theta})F^{(2)}(x)U^\dagger(x, \theta, \bar{\theta}). \quad (27)$$

Focusing on the Abelian nature of $U(x, \theta, \bar{\theta})$ and $U^\dagger(x, \theta, \bar{\theta})$, it is crystal clear that the r.h.s of (27) would yield $F^{(2)}$ only (because $UU^\dagger = U^\dagger U = I$). Thus, we have obtained the HC condition (5) for the Abelian theory (i.e. $\tilde{F}^{(2)} = F^{(2)} \iff \tilde{d}\tilde{A}_{(h)}^{(1)} = dA^{(1)}$).

The above argument and discussion can be replicated in the context of QED with complex scalar fields where, once again, we obtain the analogue of relation (22) which provides an alternative to the HC. Furthermore, we note that the following is true in the context of QED with complex scalar fields, namely;

$$DD\varphi = ieF^{(2)}\varphi, \quad (DD\varphi)^* = -ieF^{(2)}\varphi^*. \quad (28)$$

The above equation can be translated into the superfield formalism as

$$\begin{aligned} DD\varphi(x) &\rightarrow \tilde{D}\tilde{D}\Phi(x, \theta, \bar{\theta}) = ie\tilde{F}^{(2)}\Phi(x, \theta, \bar{\theta}) \equiv ie\tilde{F}^{(2)}U(x, \theta, \bar{\theta})\varphi(x) \\ (DD\varphi)^* &\rightarrow (\tilde{D}\tilde{D}\Phi)^*(x, \theta, \bar{\theta}) = -ie\tilde{F}^{(2)}\Phi^*(x, \theta, \bar{\theta}) \equiv -ie\tilde{F}^{(2)}\varphi^*(x)U^\dagger(x, \theta, \bar{\theta}), \end{aligned} \quad (29)$$

where we have: $\tilde{D}\Phi(x, \theta, \bar{\theta}) = (\tilde{d} + ie\tilde{A}_{(h)}^{(1)})\Phi(x, \theta, \bar{\theta})$ and $(\tilde{D}\Phi)^*(x, \theta, \bar{\theta}) = (\tilde{d} - ie\tilde{A}_{(h)}^{(1)})\Phi^*(x, \theta, \bar{\theta})$. With the input $\tilde{D} = UDU^\dagger$, it can be checked that

$$U(x, \theta, \bar{\theta})DD\varphi = ie\tilde{F}^{(2)}\Phi(x, \theta, \bar{\theta}) \implies \tilde{F}^{(2)} = U(x, \theta, \bar{\theta})F^{(2)}U^\dagger(x, \theta, \bar{\theta}). \quad (30)$$

The Abelian nature of $U(x, \theta, \bar{\theta})$, once again, implies that we have obtained an alternative to the HC (i.e. $\tilde{F}^{(2)} = F^{(2)}$) in the language of the SUSP unitary operator $U(x, \theta, \bar{\theta})$ because we have $U(x, \theta, \bar{\theta}) F^{(2)}(x) U^\dagger(x, \theta, \bar{\theta}) = F^{(2)}(x)$ on the r.h.s. of (30).

The celebrated HC can also be obtained from the relationship given in (22). This is due to the fact that when we operate by \tilde{d} on this equation from the left, we obtain the following explicit equation, namely;

$$\tilde{d} \tilde{A}_{(h)}^{(1)}(x, \theta, \bar{\theta}) = \tilde{d} A^{(1)}(x) - \frac{i}{e} \tilde{d} U(x, \theta, \bar{\theta}) \tilde{d} U^\dagger(x, \theta, \bar{\theta}), \quad (31)$$

where we have used (23) and the property $\tilde{d}^2 = 0$. The first term on the r.h.s. of the above equation produces: $\tilde{d} A^{(1)} = d A^{(1)} = F^{(2)}$ due to the fact that the ordinary 1-form $(A^{(1)}(x))$ connection is independent of the Grassmannian variables implying that $\partial_\theta A^{(1)} = \partial_{\bar{\theta}} A^{(1)} = 0$ when we use the definition of \tilde{d} from (6). Taking the explicit expressions for the $U(x, \theta, \bar{\theta})$ and $U^\dagger(x, \theta, \bar{\theta})$ from (15), it can be checked that $\tilde{d} U \tilde{d} U^\dagger = 0$. This statement becomes very clear if we take a close look at the exponential forms of U and U^\dagger in (16). It is obvious that the quantity in the exponent of these operators differ only by a sign factor. Thus, it can be readily checked that^{††} $\tilde{d} U \tilde{d} U^\dagger = 0$. This implies that the last term in (31) is zero (i.e. $\tilde{d} U \tilde{d} U^\dagger = 0$) which, ultimately, leads to the validity of HC (i.e. $\tilde{F}^{(2)} = F^{(2)}$). For readers' convenience, we have carried out the explicit computations of $\tilde{d} U$ and $\tilde{d} U^\dagger$ which are present now in the footnote number 6 and it can be re-checked that the second term on the r.h.s. of (31) is zero (i.e. $\tilde{d} U \tilde{d} U^\dagger = 0$).

5 Conclusions

The central objective of our present investigation has been to derive an explicit expression for the SUSP unitary operator $U(x, \theta, \bar{\theta})$ which generates the shift symmetry transformations on the gauge, (anti-)ghost and matter superfields along the Grassmannian directions of the (4, 2)-dimensional supermanifold on which the ordinary 4D interacting Abelian 1-form gauge theories (with Dirac and complex scalar fields) are generalized. In fact, the SUSP unitary operator, ultimately, leads to the derivation of proper (i.e. off-shell nilpotent and absolutely anticommuting) (anti-)BRST symmetry transformations for the above *interacting* Abelian 1-form gauge theories in physical four (3 + 1)-dimensions of spacetime.

One of the highlights of our present endeavor is the observation that the correct derivation of the SUSP unitary operator provides an alternative to the HC in addition to encompassing in its folds the sanctity of the $U(1)$ gauge group structure in the transformation space. It is the latter property which allows us to define the covariant derivative on the super matter fields (cf. (21) and (29)). This definition, in turn, leads to the derivation of a connection between the supercurvature 2-form and the ordinary curvature 2-form (cf. (26) and (29)). This is due to the fact that the commutator of two covariant derivatives defines the field strength tensor $F_{\mu\nu}$ through the relationship $[D_\mu, D_\nu] \psi = i e F_{\mu\nu} \psi$. This result has been

^{††} Using (15), it is obvious that $\tilde{d} U = dx^\mu [-i e \theta \partial_\mu \bar{C} - i e \bar{\theta} \partial_\mu C + e \theta \bar{\theta} \partial_\mu (B - e C \bar{C})] + d\theta [-i e \bar{C} + e \bar{\theta} (B - e C \bar{C})] + d\bar{\theta} [-i e C - e \theta (B - e C \bar{C})]$ and $\tilde{d} U^\dagger = dx^\nu [+i e \theta \partial_\nu \bar{C} + i e \bar{\theta} \partial_\nu C - e \theta \bar{\theta} \partial_\nu (B - e \bar{C} C)] + d\theta [+i e \bar{C} - e \bar{\theta} (B - e \bar{C} C)] + d\bar{\theta} [+i e C + e \theta (B - e \bar{C} C)]$. From these lucid expressions, one can *also* check that $\tilde{d} U \tilde{d} U^\dagger = 0$.

captured in the relationships given in eqs. (25), (26) and (28). One of the key features of our present endeavor is the observation that $U(x, \theta, \bar{\theta})$ and $U^\dagger(x, \theta, \bar{\theta})$, for both the interacting theories, turn out to be the *same*.

It would be a nice future endeavor to extend our present idea to derive explicitly the SUSP unitary operator in the context of *interacting* 4D non-Abelian 1-form gauge theory with Dirac fields which has been intelligently chosen in [1-3]. We also plan to pursue this direction of investigation in the context of *interacting* higher p -form ($p = 2, 3..$) gauge theories which are the limiting cases of (super)string theories (see, e.g. [10]). We are presently intensively involved with these issues and our future publications would resolve these in a cogent and convincing manner [11].

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